DETERMINING THE SMALLEST NUMBER OF CONTROLS NECESSARY TO STABILIZE THE EQUILIBRIUM POSITION

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The problem of determining the smallest number of controlling forces necessary to stabilize the motion of a controlled object is investigated. The necessary and sufficient condition of stabilizability of the zero solution by the control of minimal dimensionality is established for linear systems. The sufficiency criterion is extended to nonlinear systems. An example is considered.

1. Formulation of the problem. Let us consider an object whose state is described by the phase vector $x(t) = \{x_i(t)\}$ $(i = 1, \ldots, n)$ and whose motion is described by the differential system

$$dx / dt = f(x), \qquad f(0) \equiv 0$$

where f is a given n-dimensional vector function.

We assume that the object is controllable by a force $u = \{u_j\}$ $(j = 1, \ldots, r)$, where r is some number, and that the control function is variable. Let the controlling forces u_j be related to the coordinates x_i by the vector differential equation

$$dx / dt = f(x) + \varphi(x, u), \qquad \varphi(x, 0) \equiv 0 \tag{1.1}$$

where φ is some n-dimensional vector function.

Our problem consists in determining the smallest number of controlling forces necessary to stabilize the zero solution of system (1.1) with suitable choice of φ. If r is this number and φ_0 is the corresponding function, then there exists an r-dimensional control u which stabilizes the solution $x \equiv 0$ of system (1.1) for $\varphi = \varphi_0$; moreover, it is then impossible to find a function φ for which the solution $x \equiv 0$ of system (1.1) can be stabilized by means of an $(r - 1)$-dimensional control.

2. The solution of the problem in linear approximation. The linear approximation for Eq. (1.1) is of the form

$$dx / dt = Ax + Bu \tag{2.1}$$

The matrix A defines the linear operator A in the n-dimensional linear space $R \equiv \{x_1, \ldots, x_n\}$. The following theorem holds [1].

The space R can always be decomposed into the subspaces $I_1, \ldots, I_t$ cyclical relative to the given linear operator A with the minimal polynomials $\psi_1(\lambda), \ldots, \psi_t(\lambda)$,

$$R = I_1 + \ldots + I_t$$

in such a way that $\psi_1(\lambda)$ coincides with the minimal polynomial $\psi(\lambda)$ of the entire space and every $\psi_i(\lambda)$ is a divisor of $\psi_{i-1}(\lambda)$ $(i = 2, 3, \ldots, t)$.

Denoting the invariant polynomials of the matrix A by $i_1(\lambda), \ldots, i_n(\lambda)$, we find that

$$\psi_1(\lambda) = \frac{D_n(\lambda)}{D_{n-1}(\lambda)} = i_1(\lambda), \qquad \psi_2(\lambda) = \frac{D_{n-1}(\lambda)}{D_{n-2}(\lambda)} = i_2(\lambda), \ldots$$

$$\psi_t(\lambda) = \frac{D_{n-t+1}(\lambda)}{D_{n-t}(\lambda)} = i_t(\lambda), \qquad i_{t+1}(\lambda) = \ldots = i_n(\lambda) = 1$$

Here $D_i(\lambda)$ denotes the largest common divisor of all the ith order minors of the characteristic matrix $A_\lambda = A - \lambda E$, where E is an identity matrix ($i = 1, \ldots, n$).

Let us suppose that all the roots of some $D_{n-r}(\lambda)$ have negative real parts. We can show that system (2.1) can then be stabilized by an r-dimensional control with suitable choice of the matrix B.

Let b_i be the generating vector of the cyclical subspace I_i. The vectors $b_i, Ab_i, \ldots, A^{m_i-1} b_i$, where m_i is the power of $\psi_i(\lambda)$, are then linearly independent and form the basis I_i ($i = 1, \ldots, t$). As our matrix B we take the matrix

$$B = \| b_1, b_2, \ldots, b_r \| \tag{2.2}$$

and consider the matrix

$$P = \| b_1, Ab_1, \ldots, A^{m_1-1}b_1, b_2, Ab_2, \ldots, A^{m_2-1}b_2, \ldots, b_r, \ldots, A^{m_r-1}b_r, p^{(\sigma+1)}, \ldots, p^{(n)} \|$$

$$\sigma = m_1 + \ldots + m_r \tag{2.3}$$

The vectors $p^{(\sigma+1)}, \ldots, p^n$ are such that $\det P \neq 0$. The vectors p can always be chosen in an infinite number of suitable ways, since the vectors $b_1, Ab_1, \ldots, A^{m_r-1} b_r$ are linearly independent. Let us transform coordinates by means of the equation $x = Py$. In the new coordinates system (2.1) can be written as

$$y^{\cdot} = P^{-1}APy + P^{-1}Bu \tag{2.4}$$

The matrix $P^{-1}AP$ is the matrix of the operator $\mathbf{A}$ in the new basis

$$b_1, \ldots, A^{m_1-1} b_1, \ldots, A^{m_r-1} b_r, p^{(\sigma+1)}, \ldots, p^{(n)}.$$

Denote the kth vector of this basis by g_k. The kth column of the matrix $P^{-1}AP$ is filled by the coordinates of the vector Ag_k. But $Ag_k \in I_1$ for $k \leqslant m_1$, since I_1 is an invariant subspace, so that the last $n - m_1$ coordinates are equal to zero. Similarly, for $m_1 < k \leqslant m_1 + m_2$ the first m_1 and the last $n - (m_1 + m_2)$ coordinates of Ag_k are equal to zero, etc. Hence, the matrix $P^{-1}AP$ is of the form

$$P^{-1}AP = \begin{Vmatrix} L_0 & L_1 \\ 0 & L_2 \end{Vmatrix}, \qquad L_0 = \begin{Vmatrix} P_1 & 0 & \ldots & 0 \\ 0 & P_2 & \ldots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \ldots & P_r \end{Vmatrix} \tag{2.5}$$

and the characteristic polynomials of the matrices $P_1, \ldots, P_r$ are $\psi_1(\lambda), \ldots, \psi_r(\lambda)$, since the characteristic polynomial of the operator $\mathbf{A}$ in the case of cyclical subspaces coincides with the minimal polynomial of the subspace relative to this operator. The matrix

$$P^{-1}B = \| c_{ij} \| \quad (i = 1, \ldots, n;\ j = 1, \ldots, r)$$

consists of zeros and unities, which is evident from the relation $P^{-1}P = E$. Let us write out the nonzero elements c_{ij},

$$c_{11} = 1, \quad c_{m_1+1,2} = 1, \quad c_{m_1+m_2+1,3} = 1, \ldots, c_{m_1+\ldots+m_{r-1}+1,r} = 1 \tag{2.6}$$

System (2.4) breaks down into an uncontrollable part

$$dy^{(2)} / dt = L_2 y^{(2)}, \qquad y^{(2)*} = \| y_{\sigma+1}, \ldots, y_n \| \tag{2.7}$$

and a controllable part, which in turn consists of the nonhomogeneous systems

$$dy^{(j)} / dt = P_j y^{(j)} + u_j e_j + f^{(j)}(y^{(2)}) \quad (j = 1, \ldots, r) \tag{2.8}$$

$$y^{(j)*} = \| y_{m_1+\ldots m_{j-1}+1}, \ldots, y_{m_1+\ldots m_j} \|, \quad e_j^* = \| 1, 0, \ldots, 0 \|$$

Here $f^{(j)}(y^{(2)})$ is an m_j-dimensional vector function whose coordinates are linear forms of the coordinates of the vector $y^{(2)}$; the asterisk denotes transposition.

The matrix P_j corresponds to the operator A in I_j for the basis $b_j, Ab_j, \ldots, A^{m_j-1} b_j$ ($j = 1, \ldots, r$). Hence,

$$P_j = \begin{Vmatrix} 0 & 0 & 0 & \ldots & 0 & -\alpha^j_{m_j} \\ 1 & 0 & 0 & \ldots & 0 & -\alpha^j_{m_j-1} \\ 0 & 1 & 0 & \ldots & 0 & -\alpha^j_{m_j-2} \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot\cdot & \cdot\cdot\cdot\cdot \\ 0 & 0 & 0 & \ldots & 1 & -\alpha_1^j \end{Vmatrix} \qquad (2.9)$$

Here $\alpha_1^j, \ldots, \alpha^j_{m_j}$ are the coefficients of the minimal polynomial $\psi_j(\lambda)$ of the subspace I_j,

$$\psi_j(\lambda) = \lambda^{m_j} + \alpha_1^j \lambda^{m_j-1} + \cdots \alpha^j_{m_j}$$

If we set

$$u_j = \mu_1^j y_{m_1+\cdots+m_{j-1}+1} + \cdots + \mu^j_{m_j} y_{m_1+\cdots+m_j}$$

then, by suitable choice of the coefficients μ, we can ensure that the characteristic polynomial of system (2.8) has any prescribed roots with negative real parts.

Thus, any solution of initial system (2.1) can be reduced at the origin by an r-dimensional control only if the zero solution $y_{\sigma+1} = \ldots = y_n = 0$ for the uncontrollable part of system (2.7) is asymptotically stable. This is the case if and only if the characteristic roots of the matrix L_2 have negative real parts. But

$$\det \| A - \lambda E \| = \psi_1(\lambda) \ldots \psi_r(\lambda) \det \| L_2 - \lambda E_{n-\sigma} \|$$

where $E_{n-\sigma}$ is an $(n-\sigma) \times (n-\sigma)$ identity matrix.

On the other hand,

$$\psi_1(\lambda) \ldots \psi_r(\lambda) = \frac{D_n(\lambda)}{D_{n-r}(\lambda)} = \frac{\det \| A - \lambda E \|}{D_{n-r}(\lambda)}$$

Hence

$$\det \| L_2 - \lambda E_{n-\sigma} \| = D_{n-r}(\lambda) \qquad (2.10)$$

We have thus shown that if all the roots of $D_{n-r}(\lambda)$ have negative real parts, then there exists a matrix B for which asymptotic stabilization of the zero solution of system (2.1) is possible. We have assumed implicitly that $r \leqslant t$. But if $r > t$, then, since $m_1 + \ldots + m_t = n$, it follows that the zero solution can be stabilized at least by means of a t-dimensional control. This follows from the fact that the matrix L_2 does not exist in this case.

Now let us show that if $D_{n-r+1}(\lambda)$ has at least one root with a nonnegative real part, then asymptotic stabilization by a p-dimensional ($p < r$) control is impossible for any $n \times p$ matrix B_p.

Let us assume the opposite statement, i.e. that there exist vectors $b_1, \ldots, b_p$ such that the zero solution of the system

$$dx/dt = Ax + B_p u_p, \qquad B_p = \| b_1, \ldots, b_p \| \qquad (2.11)$$

is stabilizable.

The vectors $b_1, \ldots, b_p$ can be regarded as linearly independent; otherwise the control would in fact be k-dimensional, where $k < p$. Let us consider the subspace defined by the (not necessarily linearly independent) vectors

$$\begin{array}{l} b_1,\ Ab_1,\ \ldots,\ A^{n-1}b_1 \\ b_2,\ Ab_2,\ \ldots,\ A^{n-1}b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ b_p,\ Ab_p,\ \ldots,\ A^{n-1}b_p \end{array}$$

This subspace is invariant relative to the operator $\mathbf{A}$. Consequently (by virtue of the third theorem on splitting [1]) it can be split into cyclical invariant subspaces which cannot be split any further and whose minimal polynomials (which are also the characteristic polynomials) coincide with some of the factors in the expansion of $\psi_1(\lambda)$, ... $\psi_t(\lambda)$

$$\begin{array}{l} \psi_1(\lambda) = (\lambda - \lambda_1)^{c_1} (\lambda - \lambda_2)^{c_2} \ldots (\lambda - \lambda_s)^{c_s} \\ \psi_2(\lambda) = (\lambda - \lambda_1)^{d_1} (\lambda - \lambda_2)^{d_2} \ldots (\lambda - \lambda_s)^{d_s} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \psi_t(\lambda) = (\lambda - \lambda_1)^{l_1} (\lambda - \lambda_2)^{l_2} \ldots (\lambda - \lambda_s)^{l_s} \\ c_k \geqslant d_k \geqslant \ldots \geqslant l_k \qquad (k = 1, \ldots, s) \end{array}$$

Here $\lambda_1, \lambda_2, \ldots, \lambda_s$ are distinct roots of the equation $\det \| A - \lambda E \| = 0$.

Since the expansion of an invariant subspace contains "cells" with minimal polynomials which are powers of relatively prime monomials, let us say $(\lambda - \lambda_1)^{d_1}$ and $(\lambda - \lambda_2)^{c_2}$, it follows that the invariant subspace which is the direct sum of these cells is cyclical. In fact, if the generating vectors of the cells are e_1 and e_2, then the vectors

$$e_1,\ Ae_1,\ \ldots,\ A^{d_1-1}e_1,\ e_2,\ Ae_2,\ \ldots,\ A^{c_2-1}e_2$$

can be taken as the basis of the direct sum.

The matrix corresponding to the operator A in this basis is of the form

$$K_0 = \left\| \begin{matrix} K_1 & 0 \\ 0 & K_2 \end{matrix} \right\|$$

where the matrices K_1 and K_2 are of the same type as matrix (2.9).

By elementary transformations we reduce the matrix $\| K_0 - \lambda E_{d_1+c_2} \|$, where $E_{d_1+c_2}$ is a $(d_1 + c_2) \times (d_1 + c_2)$ identity matrix, to the equivalent matrix

$$\left\| \begin{matrix} K_1{}^1 & 0 \\ 0 & K_2{}^1 \end{matrix} \right\|$$

where

$$K_1{}^1 = \left\| \begin{matrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \ldots & (\lambda - \lambda_1)^{d_1} \end{matrix} \right\|, \quad K_2{}^1 = \left\| \begin{matrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \ldots & (\lambda - \lambda_2)^{c_2} \end{matrix} \right\|$$

The characteristic polynomial is $(\lambda - \lambda_1)^{d_1} (\lambda - \lambda_2)^{c_2}$, and the largest common divisor of the minors of order $(d_1 + c_2 - 1)$ is, clearly, equal to unity, since we have the minors $(\lambda - \lambda_1)^{d_1}$ and $(\lambda - \lambda_2)^{c_2}$ which are relatively prime. Hence, the characteristic polynomial of the direct sum of the cells under consideration coincides with the minimal polynomial; in other words, this direct sum is a cyclical subspace, since the degree of its minimal polynomial coincides with its dimensionality and the proposition in question has been proved.

However, if the cyclical cells have elementary divisors corresponding to the same root as their characteristic polynomials, then the invariant subspace which they form is not cyclical, i. e. it is generated by at least two vectors and their images under the operator A which are linearly independent of their originals.

In fact, let us assume that we have taken cells with the characteristic (and minimal) polynomials $(\lambda - \lambda_1)^{c_1}$ and $(\lambda - \lambda_1)^{d_1}$ $(c_1 \geqslant d_1)$ and the generating vectors a_1 and a_2, respectively. The characteristic polynomial of the direct sum of the cyclical cells under consideration is the product of $(\lambda - \lambda_1)^{c_1}$ and $(\lambda - \lambda_1)^{d_1}$. But it is clear that we can take $(\lambda - \lambda_1)^{d_1}$ as the nullifying polynomial for the whole subspace, since it nullifies both cells. However, its power is $c_1 < c_1 + d_1$, and the space cannot be cyclical, since its minimal polynomial, being a divisor of any nullifying polynomial, is of a degree smaller than the dimensionality of the subspace.

The foregoing implies that whatever the vectors $b_1, \ldots, b_p$, the characteristic polynomial $\chi(\lambda)$ of the invariant subspace defined by them and their images $Ab_1, \ldots, A^{n-1} b_1, \ldots, A^{n-1} b_p$ cannot contain more than p elementary divisors corresponding to the same root, i. e. that $\chi(\lambda)$ is a divisor of the polynomial $\psi_1(\lambda)\psi_2(\lambda)\ldots\psi_p(\lambda)$.

Let $f_1, \ldots, f_q$ be the basis vectors of the invariant subspace in question. Let us complement them to the basis throughout the space by adding the vectors $f_{q+1}, \ldots, f_n$. Finally, let us consider the matrix

$$\Phi = \| f_1, f_2, \ldots, f_q, \ldots, f_n \|$$

Let us carry out the transformation of coordinates $x = \Phi y$. System (2.11) now becomes

$$dy/dt = \Phi^{-1}A\Phi y + \Phi^{-1}B_p u_p \tag{2.12}$$

As with (2.5), the matrix $\Phi^{-1}A\Phi$ becomes

$$\Phi^{-1}A\Phi = \begin{Vmatrix} \Phi_1 & \Phi_2 \\ 0 & \Phi_3 \end{Vmatrix}$$

and system (2.12) breaks down into controllable and uncontrollable parts. The quantity Φ_3 is the matrix of the uncontrollable part. We can show that its characteristic polynomial is divisible by $D_{n-r+1}(\lambda)$, and therefore (by virtue of the above assumption concerning $D_{n-r+1}(\lambda)$) has at least one root with a nonnegative real part. This means that the uncontrollable part of the system is not asymptotically stable and proves that a p-dimensional $(p < r)$ control cannot effect asymptotic stabilization of the zero solution of system (2.1) if $D_{n-r+1}(\lambda)$ has a root with a nonnegative real part.

In fact,

$$\det \| A - \lambda E \| = \det \| \Phi^{-1}A\Phi - \lambda E \| = \psi_1(\lambda) \ldots \psi_p(\lambda) \ldots \psi_{r-1}(\lambda) \ldots$$
$$\ldots \psi_t(\lambda) = \chi(\lambda) \det \| \Phi_3 - \lambda E_{n-q} \|$$

where E_{n-q} is an $(n - q) \times (n - q)$ identity matrix. But

$$\psi_1(\lambda) \ldots \psi_{r-1}(\lambda) \ldots \psi_t(\lambda) = \psi_1(\lambda) \ldots \psi_{r-1}(\lambda) D_{n-r+1}(\lambda)$$

Since $\chi(\lambda)$ divides $\psi_1(\lambda) \ldots \psi_p(\lambda)$ (as was shown above), and since $p < r$, it follows that $\chi(\lambda)$ also divides $\psi_1(\lambda) \ldots \psi_{r-1}(\lambda)$, i. e. that

$$\det \| \Phi_3 - \lambda E_{n-q} \| = D_{n-r+1}(\lambda)\,\theta(\lambda)$$

where $\theta(\lambda)$ is some polynomial.

The above results imply the following statement.

Theorem 2.1. The zero solution of system (2.1) can be asymptotically stabilized by an r-dimensional control for some chosen matrix B and cannot be stabilized by an $(r-1)$-dimensional control for any B if and only if all the roots of the largest common divisor $D_{n-r}(\lambda)$ of the $(n-r)$-th order matrices $\| A - \lambda E \|$ have negative real parts, or if and only if $D_{n-r}(\lambda) \equiv 1$, and $D_{n-r+1}(\lambda)$ has a root with a nonnegative real part.

This theorem has been formulated for the case of asymptotic stability. If we are concerned with the stabilization of the zero solution of system (2.1) to stability only, we must require that all the roots of $D_{n-r}(\lambda)$ have nonpositive real parts, that the characteristic roots with zero real parts have simple elementary divisors only, and that $D_{n-r+1}(\lambda)$ have either a root with a positive real part or a root with a zero real part and a nonsimple elementary divisor.

The above theorem is related to [2-5] in self-evident fashion. The latter studies show that if the matrix B in (2.1) is such that under the transformation $x = Py$ the matrix L_2 of the uncontrollable part of the system in the new variables has roots with negative real parts, then it is possible to ensure asymptotic stabilization of the zero solution of (2.1) with suitable choice of the control u. Theorem 2.1 establishes the necessary and sufficient properties which the matrix A must have in order for a matrix B containing the minimum order of columns and having the above property to exist.

3. The case of a nonlinear system. Let us suppose that the function $f(x)$ in (1.1) is of the form

$$f(x) = Ax + g(x) \tag{3.1}$$

where A is an $n \times n$ matrix and $g(x)$ is the n-dimensional vector function $\{g_i(x)\}$ $(i = 1, \ldots, n)$; moreover, $g_i(x)$ begins with terms of order not lower than the second.

Theorem 3.1. The zero solution of system (1.1) for (3.1) can be asymptotically stabilized by an r-dimensional control if all the roots of the largest common divisor $D_{n-r}(\lambda)$ of the $(n-r)$-th order minors of the matrix $\| A - \lambda E \|$ have negative real parts, or if $D_{n-r}(\lambda) \equiv 1$.

In fact, let us choose the matrix B as in (2.2) and effect the transformation of variables $x = Py$, where P is the matrix defined in (2.3). Let us consider the system

$$dx/dt = Ax + g(x) + Bu \tag{3.2}$$

or, in the new variables,

$$dy/dt = P^{-1}APy + P^{-1}g(Py) + P^{-1}Bu$$

where $P^{-1}AP$ and $P^{-1}B$ are of the form (2.5), (2.6), respectively.

System (3.2) breaks down in two parts

$$dy^{(1)}/dt = L_0 y^{(2)} + L_1 y^{(2)} + C_1 u + [P^{-1}g(Py)]^{(1)}$$

$$dy^{(2)}/dt = L_2 y^{(2)} + [P^{-1}g(Py)]^{(2)} \tag{3.3}$$

where the superscript "(1)" refers to the first $(m_1 + \ldots + m_r)$ coordinates and the superscript "(2)" to the last $n - (m_1 + \ldots + m_r)$ coordinates. By C_1 we denote the matrix

$$(P^{-1}B)^* = \| C_1^*, 0 \|$$

Setting $u = My^{(1)}$, where M is an $r \times (m_1 + \ldots + m_r)$ matrix, we can, by suitable choice of M, ensure that the matrix $(L_0 + C_1 M)$ has arbitrary characteristic roots, including roots with negative real parts. Since we assumed that all of the roots of $D_{n-r}(\lambda)$ have negative real parts, it follows by (2.10) that the first approximation of (3.3) is asymptotically stable; this imples the asymptotic stability of the zero solution of system (3.2).

To complement Theorem 3.1 we can say that if $D_{n-r+1}(\lambda)$ has at least one root with a positive real part, then stabilization (not only asymptotic) by a $(r-1)$-dimensional control is impossible. The validity of this statement is self-evident from the consideration of Sect. 2.

4. Example. Let us consider a satellite in circular orbit. The equations of motion ([6], Chapter 2) are

$$J_1 p_1^{\cdot} + (J_3 - J_2)\, p_2 p_3 = 3\omega^2 (J_3 - J_2)\, \alpha_{32}\alpha_{33} \quad (1.\ 2,\ 3)$$

$$\alpha_{i1}^{\cdot} = \alpha_{i2} p_3 - \alpha_{i3} p_2 + \varepsilon_j \omega \alpha_{j1} \quad (1\ 2\ 3), \qquad (i = 1, 3) \tag{4.1}$$

$$i + j = 4, \qquad \varepsilon_j = \begin{cases} 1 & (j = 1) \\ -1 & (j = 3) \end{cases} \qquad (s=3)$$

Here J_1, J_2, J_3 are the principal central moments of inertia, $\alpha_{ik} (i = 1, 3,\ k = 1, 2, 3)$ are the relative direction cosines, and ω is the angular velocity of the center of mass along the orbit. The symbol (1 2 3) means that the two other equations are obtainable by cyclic permutation.

These equations have the particular solution

$$p_1 = p_3 = 0,\ p_2 = \omega;\ \alpha_{11} = \alpha_{33} = 1,\ \alpha_{12} = \alpha_{13} = \alpha_{31} = \alpha_{32} = 0 \tag{4.2}$$

Our task is to find the minimum number of controls required to render solution (4.2) asymptotically stable.

Taking (4.2) as the unperturbed motion and retaining the symbols for the initial variables in designating the perturbations, we can write the linear approximation for the equations of perturbed motion in the form

$$\begin{aligned}
&p_1^{\cdot} = h_{23}\omega p_3 - 3\omega^2 h_{23}\alpha_{32}, && h_{23} = (J_2 - J_3) / J_1 \\
&p_2^{\cdot} = -3\omega^2 h_{31}\alpha_{31}, && h_{31} = (J_3 - J_1) / J_2 \\
&p_3^{\cdot} = h_{12}\omega p_1, && h_{12} = (J_1 - J_2) / J_3 \\
&\alpha_{31}^{\cdot} = -p_2 + \omega\alpha_{11} - \omega\alpha_{33}, && \alpha_{11}^{\cdot} = -\omega\alpha_{13} - \omega\alpha_{31} \\
&\alpha_{32}^{\cdot} = p_1 + \omega\alpha_{12}, && \alpha_{12}^{\cdot} = -p_3 - \omega\alpha_{32} \\
&\alpha_{33}^{\cdot} = \omega\alpha_{31} + \omega\alpha_{13}, && \alpha_{13}^{\cdot} = p_2 + \omega\alpha_{11} - \omega\alpha_{33}
\end{aligned} \tag{4.3}$$

The last six equations are not independent; the α_{ik} are related by the expressions

$$(1 + \alpha_{11})^2 + \alpha_{12}^2 + \alpha_{13}^2 - 1 = 0$$

$$(1 + \alpha_{11})\, \alpha_{31} + \alpha_{12}\alpha_{32} + \alpha_{13}(1 + \alpha_{33}) = 0$$

$$\alpha_{31}^2 + \alpha_{32}^2 + (1 + \alpha_{33})^2 - 1 = 0$$

These relations enable us to determine three of the α_{ik} $(i = 1, 3,\ k = 1, 2, 3)$ as functions of the rest. For example, if the perturbations $|\alpha_{11}|, |\alpha_{33}|, |\alpha_{13}|$ are not too large, then we can take $\alpha_{12}, \alpha_{31}, \alpha_{32}$ as the independent variables.

The variables $\alpha_{13}, \alpha_{33}, \alpha_{11}$ can then be expressed in terms of the above variables, and the expansions begin with terms of higher than the second order of smallness. This enables us to take the equations

$$\begin{gathered}
p_1^{\cdot} = h_{23}\omega p_3 - 3\omega^2 h_{23}\alpha_{32}, \quad p_2^{\cdot} = -3\omega^2 h_{31}\alpha_{31}, \quad p_3^{\cdot} = h_{12}\omega p_1 \\
\alpha_{12}^{\cdot} = -p_3 - \omega\alpha_{32}, \quad \alpha_{31}^{\cdot} = -p_2, \quad \alpha_{32}^{\cdot} = p_1 + \omega\alpha_{12}
\end{gathered} \tag{4.4}$$

as the linear approximation of (4.1).

The characteristic matrix of this system is of the form

$$\left\|\begin{array}{cccccc} -\lambda & 0 & h_{23}\omega & -3\omega^2 h_{23} & 0 & 0 \\ 0 & -\lambda & 0 & 0 & -3\omega^2 h_{31} & 0 \\ h_{12} & 0 & -\lambda & 0 & 0 & 0 \\ 1 & 0 & 0 & -\lambda & 0 & \omega \\ 0 & -1 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & -1 & -\omega & 0 & -\lambda \end{array}\right\| \qquad (4.5)$$

If we assume $h_{12} = h_{23} = 0$, this matrix is equivalent to a diagonal matrix with the elements

$$a_{ij} = 0 \ (i \neq j), \qquad a_{11} = a_{22} = a_{33} = 1$$
$$a_{44} = a_{55} = \lambda, \qquad a_{66} = \lambda^4 + \omega^2\lambda^2$$

and in this case the zero solution of system (4.4) can be asymptotically stabilized by a three-dimensional control in accordance with Theorem 3.1, since $D_3 \equiv 1$.

If $h_{12} = 0$ but $h_{23} \neq 0$, the characteristic matrix is equivalent to the diagonal matrix

$$a_{ij} = 0 \ (i \neq j), \qquad a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = 1$$
$$a_{66} = \lambda^2(\lambda^2 - 3\omega^2 h_{31})(1 + \lambda^2 / \omega^2 - 3h_{31}\lambda^2)$$

and the zero solution of (4.4) can be stabilized by a single control, since $D_5 = 1$.

If $h_{12} \neq 0$ and

$$3h_{31}(3h_{31} + 3h_{23} + 1 - h_{12}h_{23}) - 4h_{12}h_{23} = 0 \qquad (4.6)$$

the characteristic matrix can be transformed to the diagonal form

$$a_{ij} = 0 \ (i \neq j), \ a_{11} = a_{22} = a_{33} = a_{44} = 1, \ a_{55} = (\lambda^2 - 3\omega^2 h_{31})$$
$$a_{66} = (\lambda^2 - 3\omega^2 h_{31})[\lambda^2 + \omega^2(3h_{23} + 1 - h_{12}h_{23} + 4\omega^2 h_{12}h_{23})]$$

and by virtue of Theorem 3.1 and the fact that $D_4 = 1$, a two-dimensional control is sufficient for asymptotic stabilization.

Finally, if $h_{12} \neq 0$ and relation (4.6) is not fulfilled, the characteristic matrix assumes the canonical form

$$a_{ij} = 0 \ (i \neq j), \ a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = 1$$
$$a_{66} = (\lambda^2 - 3\omega^2 h_{31})[\lambda^4 + \omega^2(3h_{23} + 1 - h_{12}h_{23})\lambda^2 - 4\omega^4 h_{12}h_{23}]$$

and, since $D_5 = 1$, the zero solution of system (4.4) is asymptotically stabilized by a single control in the most general case.

In practice, however, the control of minimal dimensionality can be realized only if the last three coordinates of all the vectors b of matrix (2.2) are equal to zero, since the last three equations of (4.4) are obtainable from the kinematic relations and cannot contain controls. Such choice of the vectors b is possible in all the cases considered above. We can demonstrate this directly.

For example, let $h_{12} = h_{23} = 0$. Matrix (4.5) then becomes

$$A = \left\|\begin{array}{cc} 0 & 0 \\ A_{21} & A_{22} \end{array}\right\|, \qquad A_{21} = \left\|\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array}\right\|, \qquad A_{22} = \left\|\begin{array}{ccc} 0 & 0 & \omega \\ 0 & 0 & 0 \\ -\omega & 0 & 0 \end{array}\right\|$$

In this case the stabilizing control is three-dimensional, and matrix (2.2) becomes

$$B = \| b_1 \ b_2 \ b_3 \|$$

We use the symbols $b_i^{(1)}$, $b_i^{(2)}$ to denote

$$b_i^{(1)*} = \| b_{1i} \ \ b_{2i} \ \ b_{3i} \|, \qquad b_i^{(2)*} = \| b_{4i} \ \ b_{5i} \ \ b_{6i} \|$$

Matrix (2.3) for $b_i^{(2)} = 0$ is of the form

$$P = \begin{Vmatrix} b_1^{(1)} & 0 & 0 & 0 & b_2^{(1)} & b_3^{(1)} \\ 0 & A_{21}b_1^{(1)} & A_{22}A_{21}b_1^{(1)} & A_{22}^2A_{21}b_1^{(1)} & 0 & 0 \end{Vmatrix}$$

Since the characteristic polynomial of the matrix $\|A_{22} - \lambda E\|$ coincides with its minimal polynomial, there exists an nonzero vector $c^{(1)}$ such that

$$\det \|c^{(1)} A_{22}c^{(1)} A_{22}{}^2 c^{(1)}\| \neq 0$$

Setting $A_{21}b_1^{(1)} = c^{(1)}$, $b_1^{(1)} = A_{21}^{-1} c^{(1)}$, we see that it is always possible to ensure that $\det P \neq 0$ by suitable choice of $b_2^{(1)}$ and $b_3^{(1)}$. Hence in this case B becomes

$$B = \begin{Vmatrix} b_1^{(1)} & b_2^{(1)} & b_3^{(1)} \\ 0 & 0 & 0 \end{Vmatrix}$$

If $h_{12} = 0$, $h_{23} \neq 0$, matrix (2.3) for $b^{(2)} = 0$ is of the form

$$p_{11} = l_1,\ p_{12} = h_{23}\omega l_3,\ p_{13} = -3h_{23}\omega^2 l_1,\ p_{14} - 3h_{23}\omega^3(1 - h_{23})\, l_3$$

$$p_{15} = 3h_{23}\omega^4(3h_{23} + 1)\, l_1,\quad p_{16} = 3h_{23}\omega^5(3h_{23} + 1)\,(1 - h_{23})\, l_3,\quad p_{21} = l_2$$

$$p_{23} = -3h_{23}\omega^2 l_2,\quad p_{25} = 9h_{23}{}^2\omega^4 l_2,\ p_{31} = l_3,$$

$$p_{42} = l_1,\ \ p_{43} = \omega(h_{23} - 1)\, l_3,\ \ p_{44} = -\omega^2(3h_{23} + 1)\, l_1,\ p_{45} = \omega^3\,(3h_{23} + 1)(1 - h_{23})\, l_3$$

$$p_{46} = \omega^4(3h_{23} + 1)^2\, l_1,\ p_{52} = -l_2,\ p_{54} = 3h_{23}\omega^2 l_2,\quad p_{56} = -9h_{23}{}^2\,\omega^4 l_2$$

$$p_{62} = -l_3,\ \ p_{63} = -\omega l_1,\ \ p_{64} = \omega^2(1 - h_{23})\, l_3,\quad p_{65} = \omega^3(3h_{23} + 1)\, l_1$$

$$p_{66} = \omega^5(3h_{23} + 1)(1 - h_{23})\, l_3$$

$$\det P = \pm 36\, h_{23}{}^2\omega^{12} l_2{}^2 l_3{}^2[h_{23}(h_{23} - 1)^2 l_3{}^2 + (h_{23} + 3h_{23}{}^2) l_1{}^2]$$

where only the nonzero elements have been included.

Clearly, we can always ensure that $\det P \neq 0$ by suitable choice of l_1, l_2, l_3.

All the remaining cases can be verified in similar fashion. It turns out that the control of minimal dimensionality can always be found in the form necessary for practical realization.

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BIBLIOGRAPHY

1. Gantmakher, F. R., Theory of Matrices. Moscow, "Nauka", 1966.
2 Kalman, R. E., On the general theory of control systems. In: Proceedings of the First Congress of the International Automatic Control Federation. Moscow, Izd. Akad. Nauk SSSR, Vol. 2: Theory of Discrete Optimal Self-Adjusting Systems, 1961.
3. Kirillova, F. M., Contribution to the problem on the analytical construction of regulators. PMM Vol. 25, №3, 1961.
4. Kurzweil, J., On the analytical construction of regulators. Avtomatika i Telemekhanika Vol. 22, №6, 1961.
5. Krasovskii, N. N., Theory of Motion Control. Moscow, "Nauka", 1968.
6. Beletskii, V. V., Motion of an Artificial Satellite Relative to the Center of Mass. Moscow, "Nauka", 1965.

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